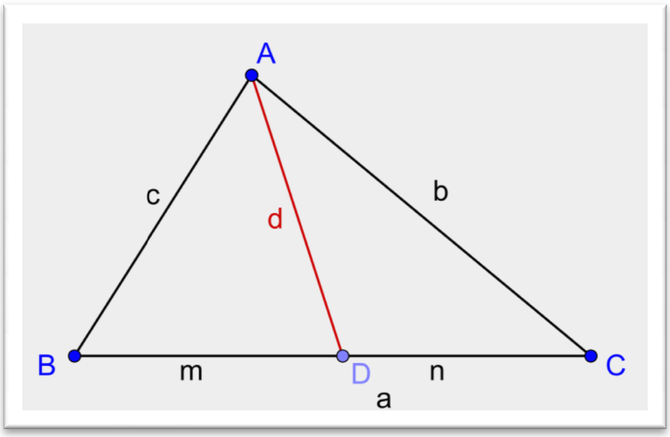
Stewart's Theorem

Stewart's Theorem, though outside the Hong Kong Exam syllabus, is one of the important theorems in geometry. It is not too difficult to learn. See whether you can manage.



Given any triangle ABC. AD is any arbitrary line. The base of the triangle: $a = m + n$.

(a) Prove Stewart's Theorem: $b^2m + c^2n = a(d^2 + mn)$ by:

(i) Pythagoras Theorem

(ii) using trigonometry.

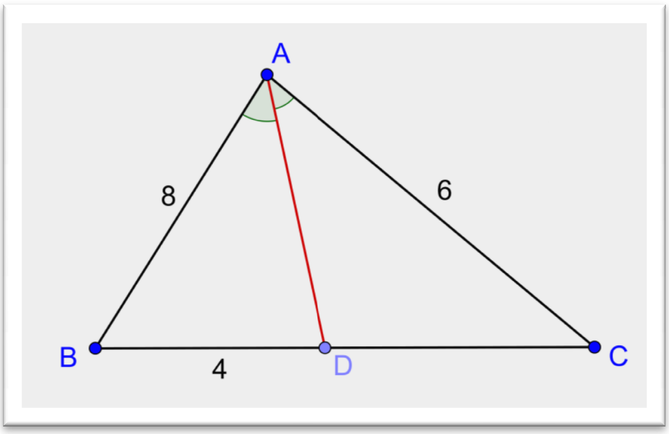
(b)

In the diagram on the right,

AD bisects $\angle BAC$.

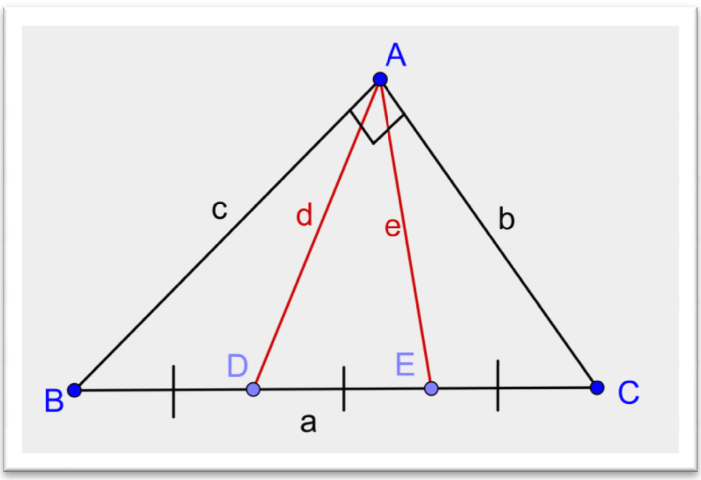
$AB = 8, AC = 6, BD = 4,$

Find the length of AD .



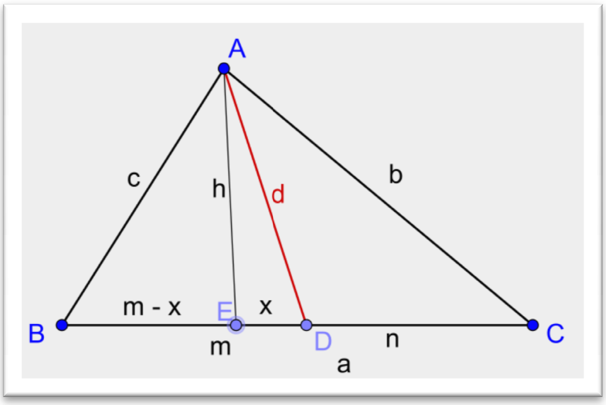
(c) Prove that the sum of the square of the distances from the vertex of the right angle, in a right angled triangle, to the trisection point of the hypotenuse, is equal to $\frac{5}{9}$ the length of the hypotenuse:

$$d^2 + e^2 = \frac{5}{9}a^2$$



(a) **(i)** Draw a perpendicular line AE from A to BC .

Let $AE = h, ED = x, BE = m - x$



Then:

$b^2 = h^2 + (n + x)^2 \quad \dots.(1)$

$c^2 = h^2 + (m - x)^2 \quad \dots.(2)$

$d^2 = h^2 + x^2 \quad \dots.(3)$

$(1) - (3), b^2 - d^2 = n^2 + 2nx \quad \dots.(4)$

$(2) - (3), c^2 - d^2 = m^2 - 2mx \quad \dots.(5)$

$(4) \times m, \quad b^2m - d^2m = n^2m + 2mnx \quad \dots.(6)$

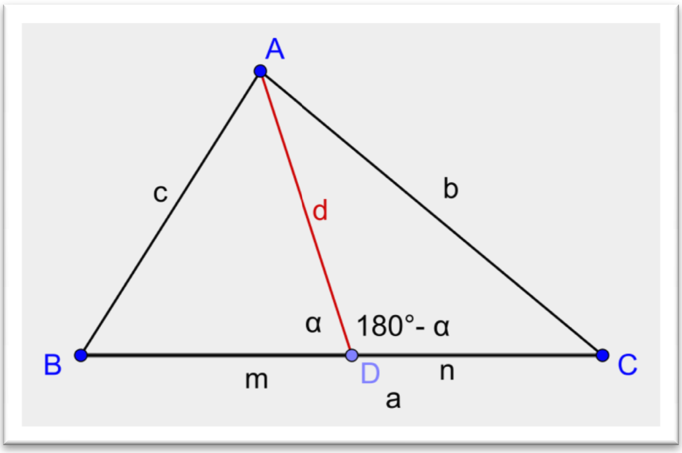
$(5) \times m, \quad c^2n - d^2n = m^2n - 2mnx \quad \dots.(7)$

$(6) + (7), \quad b^2m + c^2n - d^2(m + n) = mn(m + n)$

$b^2m + c^2n - ad^2 = amn$

$b^2m + c^2n = a(d^2 + mn)$

(ii) Let $\angle ADB = \alpha, \angle ADC = 180° - \alpha$



By Cosine Law,

$\cos\alpha = \frac{m^2+d^2-c^2}{2md} \quad \dots.(1)$

$\cos(180° - \alpha) = \frac{n^2+d^2-b^2}{2nd}$

$-\cos\ \alpha = \frac{n^2+d^2-b^2}{2nd} \quad \dots.(2)$

$\frac{m^2+d^2-c^2}{2md} = -\frac{n^2+d^2-b^2}{2nd}$

$-m(n^2 + d^2 - b^2) = n(m^2 + d^2 - c^2)$

Rearrange and putting $a = m + n$, we have

$b^2m + c^2n = a(d^2 + mn)$

Stewart's Theorem can be re-written in the form:

$man + dad = bmb + cnc$. (Mnemonic : A man and his dad put a bomb in the sink.)

(b) Let $\angle BAD = \angle CAD = \theta$

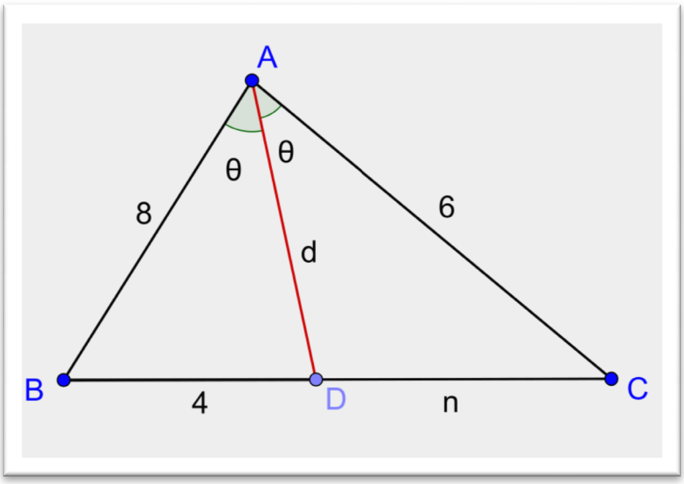
Since

$$\frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ACD} = \frac{4}{n}$$

$$\frac{\frac{1}{2}\times 8d\sin\theta}{\frac{1}{2}\times 6d\sin\theta} = \frac{4}{n}$$

$\therefore n = 3$

(This is angle bisector theorem: $\frac{8}{6} = \frac{4}{n}$.)



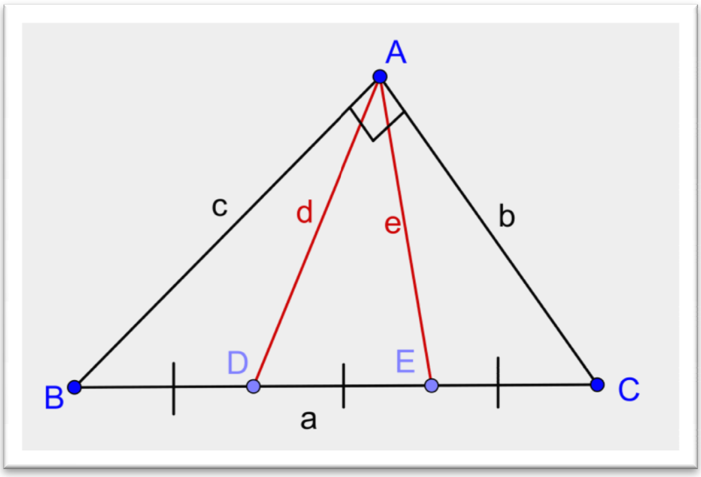
By Stewart's Theorem

$$b^2m + c^2n = a(d^2 + mn)$$

$$6^2(4) + 8^2(3) = 7(d^2 + 4 \times 3)$$

$\therefore d = 6$ (Taking positive root.)

(c) Apply Stewart theorem to ΔABC in two ways, using AD and AE as the lines of cutting the triangle:



$$\begin{cases} c^2\left(\frac{2}{3}a\right) + b^2\left(\frac{1}{3}a\right) = a\left[d^2 + \left(\frac{2}{3}a\right)\left(\frac{1}{3}a\right)\right] \\ c^2\left(\frac{1}{3}a\right) + b^2\left(\frac{2}{3}a\right) = a\left[e^2 + \left(\frac{2}{3}a\right)\left(\frac{1}{3}a\right)\right] \end{cases}$$

$$\begin{cases} c^2\left(\frac{2}{3}\right) + b^2\left(\frac{1}{3}\right) = d^2 + \left(\frac{2}{3}a\right)\left(\frac{1}{3}a\right) \\ c^2\left(\frac{1}{3}\right) + b^2\left(\frac{2}{3}\right) = e^2 + \left(\frac{2}{3}a\right)\left(\frac{1}{3}a\right) \end{cases}$$

Adding, we have $c^2 + b^2 = d^2 + e^2 + \frac{4}{9}a^2$

By Pythagoras Theorem, $a^2 = d^2 + e^2 + \frac{4}{9}a^2$

$\therefore\ d^2 + e^2 = \frac{5}{9}a^2$

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